Nonlinear viscosity derived by means of Grad's moment method

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In this paper we examine the stress tensor component evolution equations recently derived by Uribe and Garcia-Colin [Phys. Rev. E **60**, 4052 (1999)] for unidirectional flow at uniform temperature under the assumption/approximation of vanishing transversal velocity gradients. By removing this assumption/approximation we derive the stress tensor evolution equation from the Boltzmann equation within the framework of the Grad moment expansion for the case of uniform temperature (the same condition as theirs). Specializing the evolution equation to the case of steady unidirectional flow in a square channel, we obtain a set of steady state evolution equations for the components of the stress tensor. Because the transversal velocity gradients are not assumed to vanish in this paper in contrast to their paper, the present result is more general than theirs. Its special case corresponding to the one-dimensional flow considered by Uribe and Garcia-Colin is at variance with theirs because of a missing term in their stress evolution equation for the xy component. The nonlinear viscosity formulas are also different. A general remark is given with regard to the relation of dimensionalities of hydrodynamic equations and the kinetic equation underlying the former. They are not necessarily the same.

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I. INTRODUCTION

In a recent paper [1] [Phys. Rev. E **60**, 4052 (1999)], henceforth referred to as Ref. [1] in this paper, Uribe and Garcia-Colin calculate nonlinear viscosity formulas of a dilute monatomic gas that undergoes a unidirectional flow. The kinetic equation used is the Boltzmann equation and the distribution function obeying the kinetic equation is assumed to have cylindrical symmetry. The unidirectional flow is parallel to the x axis of the coordinate system. The assumed cylindrical symmetry, therefore, makes the distribution function symmetric with respect to the y and z directions. Furthermore, the temperature is assumed to be uniform so that there is no heat flow. This kind of a flow problem has been studied in the past in the dilute gas kinetic theory within the framework of the Grad moment method [2] and also in the context of generalized hydrodynamics [3] for which the dissipation terms have been in essence calculated to an infinite order in series of the Knudsen number by means of a cumulant expansion. This method for the dissipation terms ensures the nonlinear constitutive equations developed to be highly nonlinear yet still thermodynamically consistent. In the case of Ref. [1] the Grad moment method is implemented to the second order with regard to the stress tensor for the dissipation term, which is, at most, of second order in fluxes in the case of the Boltzmann equation. This feature, perhaps, distinguishes it from other works using the conventional Grad moment method for the problem or a moment method for the Maxwell model [4] or a variation [5] of the moment method. However, for a reason that is puzzling, the calculation in Ref. [1] is performed in one dimension in a rather specialized manner for a flow problem that appears in all the aspects to be a unidirectional flow in a square channel, even though the calculation can be performed in general context and more general results can be thereby obtained for the stress tensor evolution equation with no more theoretical complication than what is already incurred in their paper. In this paper, which is specifically concerned with the stress evolution equation in Ref. [1], we derive, within the framework of Grad's 13-moment expansion, the stress tensor evolution equation for a unidirectional channel flow without the assumption on the vanishing transverse velocity gradients made in Ref. [1]. The stress tensor evolution equation obtained below is, therefore, more general than theirs, and we make some deductions for nonlinear viscosities on the basis of the stress tensor evolution equation derived and compare the results with those in Ref. [1].

As is reasonably well understood in the literature [3,6,7] by now, macroscopic flow problems should be thermodynamically consistent, that is, they have to conform to the requirements of the laws of thermodynamics. Unfortunately, Grad's moment expansion method is not thermodynamically consistent as was shown in the literature [7], unless some approximations are made to the moment evolution equations obtained thereby. For example, if a perturbation solution method is applied to the moment evolution equations of Grad, the thermodynamically consistent Navier-Stokes-Fourier theory is recovered at the first order of approximation, as is the case for the first-order Chapman-Enskog solution. At the same order of approximation the local equilibrium Gibbs relation arises for the Boltzmann entropy. However, the thermodynamic consistency should hold for all orders of approximation or, better still, without any approximation, but Grad's moment method does not yield thermodynamically consistent evolution equations if no further approximation is made to them. Therefore, if one is looking for a thermodynamically consistent method for a macroscopic flow problem, there is little incentive to pursue the Grad

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moment method. Nevertheless, in Ref. [1] the aforementioned tenet of thermodynamic consistency is disregarded and the Grad method is pursued with some claims made with regards to the nonlinear longitudinal viscosity associated with the flow considered. Hence we would like to examine the evolution equations used in Ref. [1] by making calculations for a unidirectional flow, which, careful examination of the flow configuration used in Ref. [1] indicates, is equivalent to the flow configuration used in their work, and investigate under what conditions their results can be recovered and whether they are valid. In this paper we specifically consider the stress tensor evolution equation for a unidirectional flow parallel to the x axis in a square channel. The flow is assumed to be symmetric with regard to the y and z directions of the coordinate system. In Ref. [1] the assumption of vanishing transversal velocity gradients was used. In this work we remove the assumption to make our results less restricted.

In Sec. II we derive the stress tensor evolution equation in generality and then therefrom the steady state equations for various stress tensor components for the flow mentioned. It is assumed that the stress tensor has a vanishing spatial derivative, as assumed in Ref. [1]. With so-derived steady-state evolution equations, it is then possible to calculate nonlinear viscosities of various kinds. In Sec. III, the discussion and concluding remarks are given, where connection with the results of Uribe and Garcia-Colin (UGC) will also be discussed.

II. STRESS TENSOR EVOLUTION EQUATION

A. Distribution function

Because the flow is assumed to be unidirectional in the x direction and without the transversal velocity components, the mean velocity $\mathbf{u}$ of the fluid has the x component only

$$\mathbf{u}=(u_x,u_y,u_z)=(u_x,0,0). \tag{1}$$

Here we remark that the neglect of transversal velocity components is an approximation even for a channel flow from the standpoint of fluid dynamics. For this particular flow Uribe and Garcia-Colin [1] take the distribution function in the form corresponding to the ten-moment approximation of Grad

$$f(\mathbf{v},\mathbf{r},t)=f_0(1+\xi), \tag{2}$$

where

$$\xi=\mu_{xx}\left(C_x^2-\frac{k_BT}{m}\right)+\mu_{xy}(C_xv_y+C_xv_z)+\mu_{yy}\left(v_y^2+v_z^2-\frac{2k_BT}{m}\right)+\mu_{yz}v_yv_z \tag{3}$$

with the definitions $\mathbf{C}=\mathbf{v}-\mathbf{u}$ for the peculiar velocity and

$$\mu_{xx}=\frac{m}{2k_BT}\left(\frac{P_{xx}}{nk_BT}-1\right),\quad \mu_{xy}=\frac{mP_{xy}}{nk_B^2T^2},$$

$$\mu_{yy}=\frac{m}{2k_BT}\left(\frac{P_{yy}}{nk_BT}-1\right),\quad \mu_{yz}=\frac{mP_{yz}}{nk_B^2T^2}. \tag{4}$$

Here P_{xx} and so on are the Cartesian components of the stress tensor $\mathbf{P}$, k_B denotes the Boltzmann constant, and T the absolute temperature. For notational clarity the particle velocity $\mathbf{c}$ is changed to $\mathbf{v}$ in this paper. The rest of the notation is the same as in Ref. [1]. The pressure is defined, as is usual in the kinetic theory for dilute gases, by the relation $p=\frac{1}{3}\mathrm{Tr}\mathbf{P}$. The symbol Tr is the conventional notation that we usually use for trace.

The distribution function obeys the Boltzmann equation used in Ref. [1]

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\boldsymbol{\nabla}_r\right)f(\mathbf{v},\mathbf{r},t)=J(f,f), \tag{5}$$

which is a three-dimensional kinetic equation. Here $J(f,f)$ is the Boltzmann collision term, which is defined for three-dimensional collision processes of rigid spheres. Since we do not need the explicit form for this collision integral in this paper, the reader is referred to the literature [3,7–10] and Ref. [1] for its definition.

Since the Eulerian picture is taken for hydrodynamic description of flow and the kinetic theory description should be in accord with the Eulerian picture, in kinetic theory the statistical mechanical averages of microscopic variables, which are identified with hydrodynamic variables, are calculated in a coordinate system moving at the fluid velocity $\mathbf{u}$. This general viewpoint in kinetic theory is effectively implemented by introducing the peculiar velocity $\mathbf{C}$ introduced earlier. Once such a peculiar velocity is introduced and the momentum space is transformed to the moving frame, the nature of flow in the substance in question does not manifest itself in the kinetic theory calculations where the main aim is to derive the statistical mechanical formulas for material functions, such as viscosity, thermal conductivity, and diffusion coefficients, and calculate them with a suitable intermolecular potential model. To implement this program it is just necessary to derive evolution equations in the Eulerian picture for macroscopic variables involved. The important point to remember then is that the transport coefficients are calculated in the frame of reference moving at the fluid velocity, that is, in terms of the peculiar velocity introduced earlier, because the Boltzmann collision integral and related collision bracket integrals are transformed to the peculiar velocity frame [8–10] with no effect on the transport coefficients, when they are calculated in kinetic theory of material functions.

Therefore, for the purpose of our calculation here it is sufficient and convenient to put the Grad expansion for distribution function (2) in a more general form

$$f(\mathbf{v},\mathbf{r},t)=f_0\left(1+\frac{\beta}{2p}m\mathbf{CC}:\Pi\right), \tag{6}$$

where $\beta=1/k_BT$, and Π is the excess stress tensor, which in the notation used in Ref. [1] is defined by

$$\Pi=\begin{bmatrix}\Pi_{xx} & \Pi_{xy} & \Pi_{xz}\\ \Pi_{yx} & \Pi_{yy} & \Pi_{yz}\\ \Pi_{zx} & \Pi_{zy} & \Pi_{zz}\end{bmatrix}\equiv 2p(m\beta)^{-1}\begin{bmatrix}\mu_{xx} & \mu_{xy} & \mu_{xz}\\ \mu_{yx} & \mu_{yy} & \mu_{yz}\\ \mu_{zx} & \mu_{zy} & \mu_{zz}\end{bmatrix}. \tag{7}$$

The peculiar velocity in the expression of f is defined with respect to the flow defined in Eq. (1). Before proceeding to the derivation of the stress tensor evolution equation, it is helpful to observe the following.

B. Space dependence of u_x

The flow velocity is a macroscopic field variable obeying hydrodynamic (field) equations and satisfying the macroscopic initial and boundary conditions. If the flow velocity is given by Eq. (1) and is steady, as is assumed in Ref. [1], then the steady-state equation of continuity is given by

$$\boldsymbol{\nabla}_x(\rho u_x)=0, \tag{8}$$

where ρ is the mass density of the gas and $\boldsymbol{\nabla}_x=\partial/\partial x$. This equation means that

$$\rho u_x=M, \tag{9}$$

where M is independent of x and time, but may depend on coordinates y and z, since the flow is only unidirectional. Unless the gas molecules are confined to move on a line parallel to the x axis in the phase space, u_x will generally depend on y and z in the case of even the unidirectional flow considered. Since $\boldsymbol{\nabla}_x u_x$ appears in the stress tensor component evolution equations as will be shown, the stress tensor components will also depend on y and z in addition to x. Therefore, it is clear that $\boldsymbol{\nabla}_y u_x\neq 0$ and $\boldsymbol{\nabla}_z u_x\neq 0$ for the flow problem under consideration. In Ref. [1] the transversal velocity gradients are thought to be equal to zero. On the basis of the consideration given above, we consider it an assumption and will remove it; thus $\boldsymbol{\nabla}_y u_x\neq 0$ and $\boldsymbol{\nabla}_z u_x\neq 0$ in our investigation presented below.

C. Stress tensor evolution equation

On use of the Boltzmann equation, the Grad expansion (6) for the problem in hand, which is the same as the expansion in Eq. (2), and the definition of the stress tensor

$$\mathbf{P}=\int d\mathbf{v}m\mathbf{CC}f(\mathbf{v},\mathbf{r},t)=\int d\mathbf{C}m\mathbf{CC}f(\mathbf{C},\mathbf{r},t), \tag{10}$$

we obtain the general form of evolution equation for the stress tensor

$$\frac{\partial\mathbf{P}}{\partial t}=-\boldsymbol{\nabla}_{\mathbf{r}}\cdot(\psi+\mathbf{uP})-\mathbf{P}\cdot\boldsymbol{\nabla}_{\mathbf{r}}\mathbf{u}-(\boldsymbol{\nabla}_{\mathbf{r}}\mathbf{u})^t\cdot\mathbf{P}-\mu\Pi -\omega_2[\Pi\cdot\Pi]^{(2)}-\omega_1(\Pi:\Pi)\boldsymbol{\delta}. \tag{11}$$

Here various symbols are defined below: $\boldsymbol{\delta}$ is the unit second rank tensor, the symbol $[\mathbf{A}]^{(2)}$ stands for the traceless symmetric part of tensor $\mathbf{A}$, that is,

$$[\mathbf{A}]^{(2)}=\frac{1}{2}(\mathbf{A}+\mathbf{A}^t)-\frac{1}{3}\boldsymbol{\delta}\mathrm{Tr}\mathbf{A},$$

and other symbols are

$$\psi=\int d\mathbf{v}m\mathbf{CCC}f(\mathbf{v},\mathbf{r},t)=\int d\mathbf{C}m\mathbf{CCC}f(\mathbf{C},\mathbf{r},t), \tag{12}$$

$$\mu=\frac{\beta}{40p}\langle\Delta([m\mathbf{CC}]^{(2)}):\Delta([m\mathbf{CC}]^{(2)})\rangle_c, \tag{13}$$

$$\omega_2=1.5736S,\quad \omega_1=-\frac{4}{47}S. \tag{14}$$

Various symbols in these expressions are defined as below: S is defined by the Boltzmann collision bracket integral of a contracted rank-6 tensor

$$S=\frac{\beta^2}{4p^2}\langle\Delta([m\mathbf{CC}]^{(2)}):\Delta([m\mathbf{C}^*\mathbf{C}^*]^{(2)}\cdot[m\mathbf{C}_2^*\mathbf{C}_2^*]^{(2)})\rangle_c \tag{15}$$

with the subscript 2 referring to the second particle, the asterisk denoting the postcollision value. The following abbreviations are, and will be, used for the tensors involved:

$$\Delta([m\mathbf{CC}]^{(2)})=\frac{1}{4}([m\mathbf{C}^*\mathbf{C}^*]^{(2)}+[m\mathbf{C}_2^*\mathbf{C}_2^*]^{(2)}-[m\mathbf{CC}]^{(2)} -[m\mathbf{C}_2\mathbf{C}_2]^{(2)}), \tag{16}$$

$$\Delta(m\mathbf{CC})=\frac{1}{4}(m\mathbf{C}^*\mathbf{C}^*+m\mathbf{C}_2^*\mathbf{C}_2^*-m\mathbf{CC}-m\mathbf{C}_2\mathbf{C}_2), \tag{17}$$

$$\Delta([m\mathbf{C}^*\mathbf{C}^*]^{(2)}[m\mathbf{C}_2^*\mathbf{C}_2^*]^{(2)})=([m\mathbf{C}^*\mathbf{C}^*]^{(2)}[m\mathbf{C}_2^*\mathbf{C}_2^*]^{(2)} -[m\mathbf{CC}]^{(2)}[m\mathbf{C}_2\mathbf{C}_2]^{(2)}). \tag{18}$$

The dot $(\cdot)$ and double dot (:) mean the scalar product of vectors or single contraction of tensors and double contraction of tensors, respectively. Thus, for example,

$$\Delta([m\mathbf{C}^*\mathbf{C}^*]^{(2)}\cdot[m\mathbf{C}_2^*\mathbf{C}_2^*]^{(2)}) =([m\mathbf{C}^*\mathbf{C}^*]^{(2)}\cdot[m\mathbf{C}_2^*\mathbf{C}_2^*]^{(2)}-[m\mathbf{CC}]^{(2)}\cdot[m\mathbf{C}_2\mathbf{C}_2]^{(2)}).$$

The angular brackets $\langle A\rangle_c$ with the subscript c denote the collision integral defined by

$$\langle A\rangle_c=\int d\mathbf{v}\int d\mathbf{v}_2\int_0^{2\pi}d\varphi\int_0^{\infty}db\,bg_{12}f_0(\mathbf{v},\mathbf{r})f_0(\mathbf{v}_2,\mathbf{r})A.$$

This collision integral becomes that appearing in Eq. (B1) in Ref. [1], if a rigid sphere model is assumed for the interac-

tion potential. Their collision integral is clearly for three-dimensional collisions of particles. This confirms that the Boltzmann equation considered in Ref. [1] is for three-dimensional motions of particles, as pointed out earlier.

With regard to the collision bracket integrals appearing in Eqs. (13) and (15) we emphasize that they are evaluated in the moving coordinate system relative to the fluid velocity, namely, in terms of the peculiar velocities $\mathbf{C}$ and $\mathbf{C}_2$ in the collision volume moving with $\mathbf{u}$. Such mode of calculation is conventionally used in the kinetic theory of matter; see, for example, Refs. [8–10] for evaluation of the collision bracket integrals. Furthermore, collisions of monatomic gas molecules in such a moving coordinate system is independent of the nature of flow in the Boltzmann kinetic theory. Therefore, the flow does not affect the outcome of collisions of particles and thus the values of the collision bracket integrals; this means that the values of the transport coefficients remain independent of the nature of flow. This is quite reasonable, since flow should not affect the material functions of a substance. When taken in components for the flow under consideration, the quantities in Eqs. (16) and (17) can be shown to correspond to the collision integrals in Ref. [1].

The calculation of the dissipation terms in the evolution equation (11) presented earlier requires an explanation, especially, in view of the calculation made for a special flow configuration performed in Ref. [1]. If the Grad expansion (6) is inserted into the Boltzmann collision integral, the dissipation term in the evolution equation is generally given by the formula

$$\int d\mathbf{v} m\mathbf{C}\mathbf{C}J(f,f)=\mathbf{M}:\Pi+\mathbf{S}::\Pi\Pi, \tag{19}$$

where $\mathbf{M}$ is a tensor of rank 4, and $\mathbf{S}$ a tensor of rank 6, respectively, defined by the collision integrals

$$\mathbf{M}=-\frac{\beta}{2p}\langle\Delta(m\mathbf{C}\mathbf{C})\Delta([m\mathbf{C}\mathbf{C}]^{(2)})\rangle_c, \tag{20}$$

$$\mathbf{S}=-\frac{\beta^2}{4p^2}\langle\Delta(m\mathbf{C}\mathbf{C})\Delta([m\mathbf{C}^*\mathbf{C}^*]^{(2)}[m\mathbf{C}_2^*\mathbf{C}_2^*]^{(2)})\rangle_c. \tag{21}$$

The symbol (::) denotes quadruple contraction of the tensors. These tensors may be expanded into isotropic tensors [3] of rank 4 and rank 6, respectively. The calculations involved with such expansions are tedious and time consuming, but straightforward and within the limit of practicability. Calculation in fact can be programmed into a computer code, and it can be shown that they give rise to the last three terms multiplied by scalar coefficients as given in Eq. (11). Although the calculation of the numerical coefficients in Eq. (14) requires the help of a computer, the deduction of the Π dependence of the terms in fact is pretty straightforward. In view of this simple result it is puzzling why the authors of Ref. [1] chose to employ a special (i.e., one-dimensional) flow configuration and carry out the calculation component by component, when one could perform calculation in a general manner in any flow configuration in Cartesian coordinates and afterward transform the equations to other coordinate systems, if necessary; Eq. (11) holds in any flow configuration, including the one considered in Ref. [1].

In the flow configuration considered in Ref. [1] and in the Grad approximation (2), it is found that $\boldsymbol{\psi}=0$ identically. Therefore, in the approximation neglecting the derivative of the stress tensor, as is assumed in Ref. [1], the steady state stress evolution equation is generally given by

$$\begin{aligned}&\mu\Pi+\omega_2[\Pi\cdot\Pi]^{(2)}+\Pi\boldsymbol{\nabla}_\mathbf{r}\cdot\mathbf{u}+2[\Pi\cdot\boldsymbol{\nabla}_\mathbf{r}\mathbf{u}]^{(2)}\\&\quad+\boldsymbol{\delta}\left(\frac{2}{3}\mathrm{Tr}\Pi\cdot\boldsymbol{\nabla}_\mathbf{r}\mathbf{u}+\omega_1\Pi:\Pi\right)\\&\quad=-2p[\boldsymbol{\nabla}_\mathbf{r}\mathbf{u}]^{(2)}-\frac{5}{3}p\,\boldsymbol{\delta}\boldsymbol{\nabla}_\mathbf{r}\cdot\mathbf{u}.\end{aligned} \tag{22}$$

We emphasize that this equation is good for any flow configuration and for any interaction potential model.

Applying the assumed cylindrical symmetry, which also implies $\Pi_{xz}=\Pi_{xy}$, and condition (1), with definitions of the symbols

$$\gamma_x=\boldsymbol{\nabla}_x u_x, \quad \gamma_{yx}=\boldsymbol{\nabla}_y u_x, \tag{23}$$

and eliminating the $\omega_1\boldsymbol{\Pi}:\boldsymbol{\Pi}$ term between the xx and yy components of Eq. (22), we obtain the set of equations for Π_{xx},Π_{xy}, and Π_{yz} for the flow problem under consideration

$$\begin{aligned}&\begin{bmatrix}\mu+\frac{7}{3}\gamma_x & \frac{8}{3}\gamma_{yx} & 0\\ -\frac{1}{2}\gamma_{yx} & (\mu+2\gamma_x) & \gamma_{yx}\\ 0 & 0 & \mu+\gamma_x\end{bmatrix}\begin{bmatrix}\Pi_{xx}\\ \Pi_{xy}\\ \Pi_{yz}\end{bmatrix}+\omega_2\mathbf{N}(\Pi)\\&\times\begin{bmatrix}\Pi_{xx}\\ \Pi_{xy}\\ \Pi_{yz}\end{bmatrix}=-p\begin{bmatrix}\frac{4}{3}\gamma_x\\ \gamma_{yx}\\ 0\end{bmatrix},\end{aligned} \tag{24}$$

where

$$\mathbf{N}(\Pi)=\begin{bmatrix}\frac{1}{2}\Pi_{xx} & \frac{2}{3}\Pi_{xy} & \frac{2}{3}\Pi_{yz}\\ 0 & \frac{1}{2}(\Pi_{xx}+\Pi_{yz}) & \frac{1}{2}\Pi_{xy}\\ -\frac{1}{2}\Pi_{yz} & \Pi_{xy} & -\frac{1}{2}\Pi_{xx}\end{bmatrix}. \tag{25}$$

With further conditions $\boldsymbol{\nabla}_y u_x=\boldsymbol{\nabla}_z u_x=0$ as is assumed in Ref. [1], this set may be written as

$$\begin{bmatrix} \Pi_{xx} \\ \Pi_{xy} \\ \Pi_{yz} \end{bmatrix} + \omega_2 \mathbf{L}_0 \mathbf{N}(\Pi) \begin{bmatrix} \Pi_{xx} \\ \Pi_{xy} \\ \Pi_{yz} \end{bmatrix} = -\frac{4}{3} p \left(\mu + \frac{7}{3}\gamma_x \right)^{-1} \begin{bmatrix} \gamma_x \\ 0 \\ 0 \end{bmatrix}, \tag{26}$$

where

$$\mathbf{L}_0 = \begin{bmatrix} \left(\mu + \frac{7}{3}\gamma_x \right)^{-1} & 0 & 0 \\ 0 & (\mu + 2\gamma_x)^{-1} & 0 \\ 0 & 0 & (\mu + \gamma_x)^{-1} \end{bmatrix}. \tag{27}$$

This set of Eqs. (26) is comparable to the equations obtained by Uribe and Garcia-Colin for the components of the stress tensor in Ref. [1]. The following comment is useful for understanding the significance of various terms in Eqs. (24) and (26): If the channel is infinitely long, the flow velocity and other flow variables are translationally invariant along the x axis. This implies that, in particular, $\gamma_x = \nabla_x u_x = 0$. For a channel of a finite length, $\gamma_x \neq 0$ of course. But in this latter case, neither are the transversal velocity components u_y and u_z equal to zero. Consequently, the flow problem becomes much more complicated and some of the conclusions drawn on the basis of Eq. (26) or Eq. (24) become invalidated.

Equation (26) gives rise to the conclusion that $\Pi_{xy} = 0$ and $\Pi_{yz} = 0$, implying particularly that not only the shear viscosity is impossible to define in this model for the rheological constitutive equations, but also there is no shear stress in the fluid. We remark that the conditions $\boldsymbol{\nabla}_y u_x = \boldsymbol{\nabla}_z u_x = 0$ assumed for the set of Eqs. (24) are responsible for the vanishing Π_{xy} and Π_{yz}, but they are not appropriate for the unidirectional flow under consideration. We also remark that in one-dimensional shock wave study these conditions are generally used in the literature as an approximation in order to make the fluid dynamic equations one dimensional and thus as simple as possible, but they are not applied in determining the material functions of the gas used in such a shock wave study. Viscometry is usually not performed under the conditions that $\boldsymbol{\nabla}_y u_x = \boldsymbol{\nabla}_z u_x = 0$, or equivalent conditions, because it is impossible to measure a shear viscosity without shearing the fluid; for example, see Ref. [11] for some discussions on measuring viscosity.

Multiplying the inverse of the matrix on the left, we obtain from Eq. (24)

$$\begin{bmatrix} \Pi_{xx} \\ \Pi_{xy} \\ \Pi_{yz} \end{bmatrix} = -p\mathbf{L} \begin{bmatrix} \frac{4}{3}\gamma_x \\ \gamma_{yx} \\ 0 \end{bmatrix} - \omega_2 \mathbf{L}\mathbf{N}(\Pi) \begin{bmatrix} \Pi_{xx} \\ \Pi_{xy} \\ \Pi_{yz} \end{bmatrix}, \tag{28}$$

where the matrix $\mathbf{L}$ is given by

$$\mathbf{L} = \begin{bmatrix} \mu + \frac{7}{3}\gamma_x & \frac{8}{3}\gamma_{yx} & 0 \\ -\frac{1}{2}\gamma_{yx} & (\mu + 2\gamma_x) & \gamma_{yx} \\ 0 & 0 & \mu + \gamma_x \end{bmatrix}^{-1} = \begin{bmatrix} \frac{3(\mu+2\gamma_x)}{D_1} & -\frac{8\gamma_{yx}}{D_1} & \frac{8\gamma_{yx}^2}{D_2} \\ \frac{3\gamma_{yx}}{2D_1} & \frac{3\mu+7\gamma_x}{D_1} & -\frac{(3\mu+7\gamma_x)\gamma_{yx}}{D_2} \\ 0 & 0 & \frac{1}{\mu+\gamma_x} \end{bmatrix} \tag{29}$$

with the definitions

$$D_1 = 3\mu^2 + 13\mu\gamma_x + 4\gamma_{yx}^2 + 14\gamma_x^2,$$
$$D_2 = 3\mu^3 + 16\mu^2\gamma_x + 27\mu\gamma_x^2 + 14\gamma_x^3 + 4\gamma_{yx}^2\mu + 4\gamma_{yx}^2\gamma_x. \tag{30}$$

Equation (28) is not solvable in a simple closed form, but may be iteratively solved, by treating ω_2 as the expansion parameter, just to gain useful insights into the stress tensor components in the case of the magnitude of ω_2 is small. (It can be shown that ω_2 is proportional to $N_{Kn}^{\varepsilon}(\varepsilon > 0)$, where N_{Kn} is the Knudsen number, which is much smaller than unity in the case of gases at a normal state.)

To the lowest order, neglecting the terms containing ω_2, we obtain from Eq. (28)

$$\begin{bmatrix} \Pi_{xx} \\ \Pi_{xy} \\ \Pi_{yz} \end{bmatrix} = -p\mathbf{L} \begin{bmatrix} \frac{4}{3}\gamma_x \\ \gamma_{yx} \\ 0 \end{bmatrix}. \tag{31}$$

To this order of approximation, we therefore find the three stress tensor components in the forms

$$\Pi_{xx} = -\frac{p}{D_1}[4(\mu + 2\gamma_x)\gamma_x - 8\gamma_{yx}^2], \tag{32}$$

$$\Pi_{xy} = -\frac{3p}{D_1}(\mu + 3\gamma_x)\gamma_{yx}, \tag{33}$$

$$\Pi_{yz} = 0. \tag{34}$$

The presence of γ_{yx} in these equations for the stress components is easy to understand because the unidirectional flow creates a velocity (shear) gradient in the transversal direction owing to the flow stagnating (or sticking) at the boundaries, as is well known. Since $\gamma_{yx} = \boldsymbol{\nabla}_y u_x = 0$ in Ref. [1], it follows $\Pi_{xy} = 0$, yielding the conclusion that the shear viscosity is not defined, but the shear viscosity of the gas is well defined for the flow configuration. Any rheological constitutive equation that suggests an undefined shear viscosity while there is a longitudinal viscosity should be a suspect, because $\gamma_x = 0$ if

the channel is infinitely long, as mentioned earlier, and consequently $\Pi_{xx}=\Pi_{xy}=\Pi_{yz}=0$. This conclusion clearly is contrary to experiment and the nature of viscous flow. In Ref. [1] it is claimed to be a novelty to viscous phenomena.

These results in Eqs. (32) and (33) suggest that if the longitudinal viscosity is defined by the relation

$$\Pi_{xx}=-\eta_l\gamma_x,$$

then the nonlinear longitudinal viscosity is given by the formula

$$\eta_l=\frac{4p}{D_1}\left[(\mu+2\gamma_x)-8\frac{\gamma_{yx}^2}{\gamma_x}\right]. \tag{35}$$

In rheology it is the usual practice to define normal stress differences and the corresponding normal stress difference coefficients. There are two normal stress differences, primary and secondary. According to the convention used in rheology [11,12], the primary normal stress difference is defined for the flow under consideration by the difference

$$N_1=\Pi_{xx}-\Pi_{yy}, \tag{36}$$

whereas the secondary normal stress difference is defined by

$$N_2=\Pi_{yy}-\Pi_{zz}. \tag{37}$$

By the symmetry assumed for the flow the secondary normal stress difference is equal to zero because

$$\Pi_{yy}=\Pi_{zz}.$$

This also implies that

$$\Pi_{xx}=-2\Pi_{yy},$$

as was observed by Uribe and Garcia-Colin. Consequently, the primary normal stress difference is given by

$$N_1=\frac{3}{2}\Pi_{xx}. \tag{38}$$

In rheology the primary normal stress difference coefficient η_x is defined by the relation

$$N_1=-\eta_x\gamma_{yx}^2. \tag{39}$$

Therefore, to the order of approximation for Π_{xx} given in Eq. (32), the primary normal stress difference coefficient is given by

$$\eta_x=-\frac{3p}{2D_1}\left[4\mu\frac{\gamma_x}{\gamma_{yx}^2}+8\left(\frac{\gamma_x^2}{\gamma_{yx}^2}-1\right)\right]. \tag{40}$$

Since the shear viscosity is commonly defined by the relation

$$\Pi_{xy}=-\eta_s\gamma_{yx} \tag{41}$$

in the tradition of Newton, the nonlinear shear viscosity for the approximation used for Π_{xy} in Eq. (33) is given by the formula

$$\eta_s=\frac{3p}{D_1}(\mu+3\gamma_x). \tag{42}$$

For the flow considered in Ref. [1] η_s is not defined because of the conditions $\nabla_y u_x=\nabla_z u_x=0$. In any case, in Ref. [1] the normal stress differences are not considered. These formulas also give different viscosity values depending on the sign of γ_x, although they are all even with respect to γ_{yx}. The odd powered γ_x dependence of the viscosity is also present in the UGC result. The odd powered γ_x dependence is suggested as the saving grace of their result by Uribe and Garcia-Colin, but it must be recognized that such a dependence on γ_x produces two different sets of nonlinear material functions that are different for the compression and decompression of the gas. This is a peculiar feature of their result; it in fact should be regarded as a defect of the theory or at least as an indicator that there is something strange in the constitutive model for the stress tensor components. The same feature is also present in the papers by Santos [4] and Karlin *et al.* [5], although the methods used are different. Since the latter authors use a unidirectional flow, which does not depend on *y* and *z*, the underlying cause for the similar results probably is the same as for Ref. [1], namely, a unidirectional flow in which u_x is a function of *x* only. In any case, since the aim of discussion in this paper is not in the papers of Santos and Karlin *et al.*, we do not dwell on their results in this work.

The present lowest-order iterative results (32) and (32) suggests

$$\Pi_{xx}\to -p, \quad \Pi_{xy}\to 0 \quad \text{as} \quad |\gamma_x|\to\infty,$$

although the approach to the limits has two different paths depending on the sign of γ_x.

In the large $|\gamma_x|$ limit the lowest-order iterative solution presented above is inadequate, of course, and one must numerically solve the equations in Eq. (28). It is worth noting that in the case of an infinitely long channel, the flow is translationally invariant as noted earlier. This means that u_x is independent of *x*, although it depends on *y* and *z*. In this case, we have $\gamma_x=0$ and the nonlinear viscosity depends on the shear rate γ_{yx} only. For this flow the longitudinal viscosity is not defined, but the approximate stress components in Eqs. (32) and (14) predict the nonlinear shear viscosity and the primary normal stress coefficients in the forms

$$\eta_x=\frac{12p\mu}{3\mu^2+4\gamma_{yx,}^2}, \tag{43}$$

$$\eta_s=\frac{3p\mu}{3\mu^2+4\gamma_{yx}^2}, \tag{44}$$

both of which vanish as $|\gamma_{yx}|\to\infty$. However, it must be noted that $\Pi_{xx}\to -3p\mu$ whereas $\Pi_{xy}\to 0$ as $|\gamma_{yx}|\to\infty$. In this approximation the behavior of the shear stress is unphysical in the high shear limit, because it cannot vanish in such a limit. It seems to suggest that the underlying stress evolution equation, namely, the constitutive equation, is inadequate.

The fact that $\Pi_{yz}=0$ implies that the fluid does not rotate within the square cross section of the duct, as it should not owing to the symmetry assumed. This result is physically reasonable. However, in the higher-order approximation $\Pi_{yz}\neq 0$, which means that there can arise secondary flows in the duct, and such secondary flows probably occur at the corners of the square cross section. The results obtained by Uribe and Garcia-Colin in Ref. [1] are at variance with the results deduced from the evolution equations (28), which are general for the flow configuration considered.

It must be emphasized that $\Pi_{xy}\neq 0$ in general in contrast to the conclusion of Ref. [1] that $\Pi_{xy}=0$, which is indeed strange for a duct flow because it does not take the viscous effect into account and consequently contradicts the Poiseuille flow profile that is well known in hydrodynamics and used for measuring shear viscosity. This variance arises from their incorrect treatment of the stress evolution equation by setting $\nabla_y u_x=\nabla_z u_x=0$ from the outset.

III. DISCUSSION AND CONCLUDING REMARKS

With regard to the relation of the dimensionalities of the kinetic equation and a flow problem (i.e., hydrodynamic equations), we make the following general remark, not necessarily confined to Ref. [1], on the use of the Boltzmann equation in the literature. It is general practice in the dilute gas kinetic theory [2,8,10] that the configurational part of the phase space $(m\mathbf{v},\mathbf{r})$, where the Boltzmann equation lives, is used coincidently with the configuration space $\mathbf{r}$ of the hydrodynamic equations. In other words, the position variable $\mathbf{r}$ in the Boltzmann equation has the same notation and thus the same mathematical meaning as the position variable $\mathbf{r}$ in the macroscopic equations (e.g., the moment equations or the Navier-Stokes-Fourier equations) derived from the former. However, the two configuration spaces are physically not the same, although they appear to be so at first glance. It must be recalled that the kinetic theory yields a mathematical model for description of the macroscopic variables of the fluid particle of an elementary volume that contains a large number of molecules. (Note that a fluid particle is not the same as a molecule.) Therefore, even if the fluid particle moves one dimensionally in its hydrodynamic configuration space, it does not mean that *the molecules making up the fluid particle and contained in the elementary volume of the hydrodynamic configuration space*, which should be larger than the molecular collision volume and over which the statistical averaging is performed, *should be moving one dimensionally*. Since the configuration space of the Boltzmann equation in Eq. (5) is made coincident with the hydrodynamic configuration space as a mathematical shortcut in the conventional treatment of gas kinetic theory, one might be misled that the Boltzmann equation should be also one dimensional if the hydrodynamic motion is one dimensional. Unfortunately, this misleading form is commonly taken for the kinetic equation in gas kinetic theory and also in Ref. [1].

This confusing feature of Eq. (5) can be removed if the phase space and the hydrodynamic configuration space are clearly distinguished. This aim can be easily achieved if we define the Boltzmann equation in the phase space, say, $(\mathbf{p}_i,\mathbf{r}_i)$—the μ space of particle i—where $\mathbf{p}_i=m\mathbf{v}_i$ and $\mathbf{r}_i$ is the position vector of the ith molecule of N molecules contained in volume V

$$\left(\frac{\partial}{\partial t}+\mathbf{v}_i\cdot\frac{\partial}{\partial \mathbf{r}_i}\right)F_i(\mathbf{p}_i,\mathbf{r}_i,t)=\sum_{j=1}^{N} J(F_i,F_j), \tag{45}$$

and if the mean hydrodynamic variable $A(\mathbf{r},t)$ corresponding to dynamical variable $A_i(\mathbf{p}_i,\mathbf{r}_i)$ is defined by

$$A(\mathbf{r},t)=V^{-1}\sum_{i=1}^{N}\int d\mathbf{p}_i\int_V d\mathbf{r}_i A_i(\mathbf{p}_i,\mathbf{r}_i)\,\delta(\mathbf{r}_i-\mathbf{r})F_i(\mathbf{p}_i,\mathbf{r}_i,t). \tag{46}$$

The singlet distribution functions $F_i(\mathbf{p}_i,\mathbf{r}_i,t)$ are normalized as follows:

$$\int d\mathbf{p}_i\int_V d\mathbf{r}_i F_i(\mathbf{p}_i,\mathbf{r}_i,t)=1. \tag{47}$$

With Eqs. (45)–(47) it is possible to obtain exactly the same hydrodynamic equations as with the Boltzmann equation (5), but the aforementioned confusion regarding the configuration spaces of the μ space and the hydrodynamic variables can be avoided therewith. It clearly shows that the kinetic equation is always three dimensional in the configuration part of the phase μ space regardless of the dimensionality of the hydrodynamic motion because the hydrodynamic variables are defined in a coarser scale in the configuration space than the molecular variables defined in the fine-grained phase space (i.e., μ space) and are mean values exhibited by a group of molecules contained in an elementary volume of the hydrodynamic configuration space. We note that the mean values for dense correlated particles are in fact defined in the same mode as in Eq. (46) in the kinetic theory of dense gases and liquids [7,10]. The presumption of $\nabla_y u_x=\nabla_z u_x=0$ in Ref. [1] can be deemed to be a result of not distinguishing the configuration spaces in kinetic theory of gases and hydrodynamics.

An important general defect of the distribution function in Eq. (2) is that it does not give rise to a continuum hydrodynamic theory of stress phenomena that is consistent with the thermodynamic laws, as previously pointed out by the present author [7]. Therefore, neither can the nonlinear viscosity calculated therewith be expected to be consistent with the thermodynamic laws. It is our experience that when material functions are not consistent with the thermodynamic laws they do not behave properly, at least, in some aspects. Therefore, the calculation of the nonlinear longitudinal viscosity by Uribe and Garcia-Colin [1], who ignore the question of thermodynamic consistency, seems to be a throwback to the days when the question of thermodynamic consistency of material functions did not draw attention.

To understand the point of Eq. (24) by using the procedure taken in Ref. [1] we return to Eq. (A2) of Ref. [1]. In the present notation it is given by

$$\int d\mathbf{v} m C_x^2\mathbf{v}\cdot\nabla_r f=m\int d\mathbf{v} C_x^2 v_x\frac{\partial}{\partial x}f.$$

For the second line in this equation the distribution function in Eqs. (2) and (3) is used. Since the Boltzmann equation is not one dimensional and neither are particle motions one dimensional in view of the distribution function taken in Eq. (2), nor are the macroscopic variables independent of y and z, it is incorrect to replace $\mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{r}}$ on the left with $v_x\partial/\partial x$. Let us calculate the left-hand side in components. We obtain

$$\int d\mathbf{v} m C_x^2 \mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{r}} f = \int d\mathbf{v} m C_x^2 (v_x\boldsymbol{\nabla}_x + v_y\boldsymbol{\nabla}_y + v_z\boldsymbol{\nabla}_z) f$$
$$= 2\boldsymbol{\nabla}_x (Q_x + u_x P_{xx}) + 2P_{xx}\boldsymbol{\nabla}_x u_x + 2P_{yx}\boldsymbol{\nabla}_y u_x + 2P_{zx}\boldsymbol{\nabla}_z u_x .$$

In Eq. (A2) of Ref. [1] Q_x should read $\boldsymbol{\nabla}_x Q_x$ and the last two terms on the right in the equation above are missing because it is assumed that $\boldsymbol{\nabla}_y u_x = \boldsymbol{\nabla}_z u_x = 0$. This clearly demonstrates that the xx component equation used in Ref. [1] for the stress tensor is inappropriate for the problem in question since setting $\boldsymbol{\nabla}_y u_x = 0$ is inconsistent with the Grad approximation made for the distribution function, with the equation of continuity for the flow configuration considered, and with the stress evolution equations derived for P_{yy}, P_{xy}, and P_{yz}, etc. in Ref. [1]. Note in this regard that $\boldsymbol{\nabla}_x u_x = M(y,z)\boldsymbol{\nabla}_x v$ as pointed out earlier, and it implies the stress tensor can depend on y and z. The origin of the absence of the term $\boldsymbol{\nabla}_y u_x$ in some of the evolution equations for the stress components in their paper is evidently in the incorrect use of the kinetic equation when the stress evolution equation is derived, as we have earlier shown explicitly with Eq. (A2) in Appendix A of Ref. [1]. For the reason mentioned earlier, the distribution function f used is not a function of x only, if the flow is unidirectional. The same error persists in the evolution equations for other components of the stress tensor because of the erroneous assumption on the y and z dependence of u_x.

In conclusion, in this work we have derived, by using the Grad expansion for a unidirectional channel flow at a uniform temperature, the stress tensor evolution equation in general form. The steady state form of this equation is then used to derive the constitutive equations for the stress tensor components Π_{xx}, Π_{xy}, and Π_{yz}. The nonlinear viscosity formulas associated with these stress tensor components can be obtained if the nonlinear constitutive equations are solved. Since the velocity gradients $\boldsymbol{\nabla}_y u_x$ and $\boldsymbol{\nabla}_z u_x$ do not vanish in general in the case of the unidirectional flow considered, the aforementioned constitutive equations are at variance with the corresponding equations obtained in Ref. [1]. Because u_x is assumed to be a function of x only in Ref. [1] and this in turn gives rise to $\boldsymbol{\nabla}_y u_x = \boldsymbol{\nabla}_z u_x = 0$, the terms related to the velocity gradients $\boldsymbol{\nabla}_y u_x$ and $\boldsymbol{\nabla}_z u_x$ are unjustifiably absent in the stress evolution equations derived by Uribe and Garcia-Colin, making them inappropriate for the unidirectional flow. In any event, the stress evolution Eq. (11) is not thermodynamically consistent, as was pointed out in the literature [7], if the Grad approximation for the distribution function is used, and this weakness makes the nonlinear longitudinal viscosity computed from the stress evolution equations poorly behaved in some aspects and consequently ill suited for flow studies in the nonlinear regime.

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